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Mass-Energy in Static Gravitational Fields

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The mass-energy equation in static gravitational fields is shown to be $E = g_{44}mc^2$, which agrees with the expression $E = m_0c^2(dx^\mu/ds)\xi_\mu$ for the energy in a gravitational field possessing a timelike Killing vector ξ. For the Schwarzschild field this leads to $E_s \cong m_0c^2 + \frac{1}{2}m_0v^2 - km_0M/r$. For the Reissner–Nordström field an additional term describing the interaction between the mass and the charge is found to be $2\pi km_0Q^2/c^2r^2$. In the Kerr–Newman case more terms are found due to the central rotating gravitating mass.

1. INTRODUCTION

The famous mass-energy equation of Einstein, $E = mc^2$, in the special theory of relativity is associated with force F in the following manner:

$$W = \int_{p_0}^{p} F\,dx = \int_{p_0}^{p} \frac{d}{dt}\left(m\frac{dx}{dt}\right)dx = mc^2 - m_0c^2 \tag{1}$$

$$E = W + m_0c^2 = mc^2 \tag{2}$$

where the force acts on the particle to accelerate it. However, according to the general theory of relativity, a gravitating system must be described in terms of a Riemannian geometry defined by the metric tensor $g_{\mu\nu}$. The expression of F must then be so generalized to accommodate the $g_{\mu\nu}$ and their derivatives as to properly fit in with general relativity. The mass-energy equation thus obtained may therefore contain $g_{\mu\nu}$ or their derivatives as its components. Equation (2) in flat space is the limiting case of this general mass-energy equation. Moreover, since gravity is a metric phenomenon in general relativity, it may also be expected that the classical gravitational potential energy will be embedded in this general mass-energy equation.

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2. NONVANISHING MINKOWSKI ACCELERATION VECTOR

The gravitational field equations for nonempty space are

$$G^{\mu\nu}=R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R=\frac{-8\pi k}{c^2}T^{\mu\nu} \tag{3}$$

with $T^{\mu\nu}$ denoting the energy-momentum tensor of all kinds of energy content in the space except the gravitational one. When only the noninteracting incoherent matter is involved, the energy-momentum tensor is merely the matter tensor:

$$T^{\mu\nu}=M^{\mu\nu}=\rho_0\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{4}$$

The trajectory of a material point in this matter field is then determined by the geodesic equations

$$T^{\mu\nu};\nu=M^{\mu\nu};\nu=\rho_0 f^\mu=0 \tag{5}$$

$$f^\mu=\frac{D}{ds}\left(\frac{dx^\mu}{ds}\right)=\frac{d^2x^\mu}{ds^2}+\left\{\begin{matrix}\mu\\ \alpha\beta\end{matrix}\right\}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \tag{6}$$

where D/ds is the covariant derivative with respect to the affine parameter s, and the f^μ is called the Minkowski acceleration vector. If, however, other kinds of force field are present, they must be included in the energy-momentum tensor $T^{\mu\nu}$ to make it complete. Setting the divergence of the complete energy-momentum tensor equal to zero will give the correct equations of motion. Denoting all kinds of nongravitational field other than the matter field by $S^{\mu\nu}$, we have

$$T^{\mu\nu}=M^{\mu\nu}+S^{\mu\nu} \tag{7}$$

and, as a consequence of this,

$$f^\mu=-\frac{1}{\rho_0}S^{\mu\nu};\nu \tag{8}$$

A nonvanishing $S^{\mu\nu}$ will not have zero divergence in general since the $S^{\mu\nu}$ is not a complete energy-momentum tensor in itself. Hence the trajectory of a material particle in the presence of $S^{\mu\nu}$ field is not, generally speaking, given by the geodesic equations (5). It is, instead, given by (8). The particle is thus subject to the effect of $S^{\mu\nu}$ field, and accelerated.

3. $E = g_{44}mc^2$ IN THE STATIC GRAVITATIONAL FIELDS

First of all let us take into account only the static gravitational fields

$$
\begin{aligned}
&ds^2 = g_{44}(dx^4)^2 - g_{ij}dx^i dx^j \\
&i, j = 1, 2, 3 \\
&g_{4i} = g_{i4} = 0 \\
&x^4 = ct
\end{aligned}
\tag{9}
$$

g_{44} and g_{ij} being independent of x^4. Dividing the above equation by ds^2 and taking the covariant derivatives with respect to s over both sides, we get

$$g_{44} f^4 dx^4 - g_{ij} f^i dx^j = 0 \tag{10}$$

where f^4 and f^i are given by equation (6). The f^4 in equation (10) can be further reduced as follows. From the Christoffel 3-index symbols of the second kind

$$\begin{Bmatrix} 4 \\ 44 \end{Bmatrix} = 0, \qquad \begin{Bmatrix} 4 \\ ij \end{Bmatrix} = 0, \qquad \begin{Bmatrix} 4 \\ 4i \end{Bmatrix} = \frac{1}{2} g^{44} \frac{\partial g_{44}}{\partial x^i}$$

we have, by virtue of equation (6),

$$
\begin{aligned}
f^4 &= \frac{d^2x^4}{ds^2} + g^{44} \frac{\partial g_{44}}{\partial x^i} \frac{dx^i}{ds} \frac{dx^4}{ds} \\
&= \frac{1}{g_{44}} \frac{d}{ds} \left(g_{44} \frac{dx^4}{ds} \right)
\end{aligned}
\tag{11}
$$

Classically the force is defined by

$$F = \frac{d}{dt} \left(m \frac{dx}{dt} \right) \tag{12}$$

In general relativity we must generalize F to include the metric tensor $g_{\mu\nu}$, as stated previously. The simplest way is to replace the ordinary derivative before the parenthesis in equation (12) by the covariant derivative (Sokolnikoff, 1969)

$$F^j = \frac{D}{dt} \left(m \frac{dx^j}{dt} \right), \qquad m = m_0 \frac{dx^4}{ds} \tag{13}$$

This equation can be put in a more convenient form which reads

$$F^j = m_0 c^2 \left(\frac{dx^4}{ds} \right)^{-1} f^j \tag{14}$$

The work done by the generalized force F^j in displacing a particle by a distance dx^j in the $S^{\mu\nu}$ field is therefore

$$W = \int_{p_1}^{p_2} F_j\, dx^j = \int_{p_1}^{p_2} g_{ij} F^i\, dx^j$$

which becomes, using equations (10), (11), and (14),

$$W = \int_{t_1}^{t_2} m_0 c^2 \frac{ds}{dx^4} g_{44} f^4\, dx^4 = m_0 c^2 \int_{t_1}^{t_2} d\left(g_{44} \frac{dx^4}{ds} \right)$$

Thus

$$W = m_0 c^2 g_{44} \frac{dx^4}{ds} + \text{const} \tag{15}$$

The constant of integration can be determined by noting the fact that $f^j = 0$, $F^j = 0$ and consequently $W = 0$ when the $S^{\mu\nu}$ field vanishes. Hence

$$\text{const} = -m_0 c^2 g_{44} \frac{dx^4}{ds} \bigg|_{S^{\mu\nu} = 0} \tag{16}$$

Substitution back into equation (15) gives

$$W = m_0 c^2 g_{44} \frac{dx^4}{ds} \bigg|_{S^{\mu\nu} \neq 0} - m_0 c^2 g_{44} \frac{dx^4}{ds} \bigg|_{S^{\mu\nu} = 0} \tag{17}$$

The lost term in equation (17) is the energy of the test particle at infinity:

$$E_0 = m_0 c^2 g_{44} \frac{dx^4}{ds} \bigg|_{S^{\mu\nu} = 0} \tag{18}$$

As the particle proceeds towards the center of the $S^{\mu\nu}$ field, it gains an amount of energy W to arrive at a total energy

$$E = E_0 + W = m_0 c^2 g_{44} \frac{dx^4}{ds} \bigg|_{S^{\mu\nu} \neq 0} \tag{19}$$

In view of the nearly identical expression of equations (18) and (19), and the fact that the $S^{\mu\nu}$ field is arbitrary, we can write the total energy of the particle in any static field as

$$E = m_0 c^2 g_{44} \frac{dx^4}{ds} = g_{44} m c^2 \tag{20}$$

whether the $S^{\mu\nu}$ field is present or not. Being a function of space coordinates, the expression of $g_{44}\dfrac{dx^4}{ds}$ changes as the particle moves towards or away from a force field, and the total energy content of that particle changes accordingly. In flat space, equation (20) is evidently the mass-energy equation of Einstein in special relativity.

4. $E = g_{44} mc^2$ EXPANDED IN TAYLOR SERIES

To get a closer insight into equation (20) we try to expand it in terms of the Taylor series, and see what meaning each term will have from the view-point of classical mechanics.

From the metric (9) we get

$$\left(\frac{dx^4}{ds}\right)^2 = \frac{1}{g_{44} - v^2/c^2}, \qquad v^2 = g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} \tag{21}$$

Taking the square root and applying it to the mass-energy equation (20), we obtain

$$E = m_0 c^2 \frac{g_{44}}{\left(g_{44} - v^2/c^2\right)^{1/2}} \tag{22}$$

where the positive root is adopted since we want this equation to reduce to that of special relativity for large radial distances $g_{44} \approx 1$. The two-dimensional Taylor expansion of the function $\Gamma(X,Y)$ around the point $(X_0 = 1, Y_0 = 0)$ is

$$\Gamma(X,Y) = \frac{X}{(X-Y)^{1/2}} = 1 + \frac{1}{2}(X-1) + \frac{1}{2}Y + \cdots \tag{23}$$

We have by identifying g_{44} with X and v^2/c^2 with Y, and by using equations (22) and (23),

$$E \cong m_0 c^2 + \tfrac{1}{2} m_0 v^2 + \tfrac{1}{2} m_0 c^2 (g_{44} - 1) \tag{24}$$

for low velocity. The Schwarzschild metric is of the standard form (9); equations (20) and (24) are therefore directly applicable to it. This yields

$$E_S \cong m_0c^2 + \frac{1}{2}m_0v^2 - \frac{km_0M}{r} \tag{25}$$

in which the subscript S stands for the Schwarzschild field. The first and second terms in (25) are the rest and kinetic energies of the test particle, respectively. The third term is the familiar gravitational potential energy of Newton.

In the field of Reissner (1916) and Nordström (1918), the g_{44} is given by

$$g_{44} = 1 - \frac{2kM}{rc^2} + \frac{4\pi kQ^2}{c^4r^2} \tag{26}$$

So the total energy of the test particle in this field is

$$E_{R-N} \cong E_S + \frac{2\pi km_0Q^2}{c^2r^2} \tag{27}$$

where the E_S is given by equation (25). The additional term in equation (27) is due to the presence of an electrostatic field; it may therefore be called the mass–charge interaction energy between the test particle and the source.

As to the Kerr–Newman metric (Kerr, 1963; Newman et al., 1965) expresses in terms of Boyer–Lindquist (1967) coordinates

$$\begin{aligned} ds^2 = {} & \left(1 - \frac{\Omega}{\Sigma}\right)c^2\,dt^2 - \frac{2a\Omega}{\Sigma}\sin^2\theta\,d\varphi\,cdt \\ & - \frac{A}{\Sigma}\sin^2\theta\,d\varphi^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma\,d\theta^2 \end{aligned} \tag{28}$$

with the following abbreviations:

$$\Omega = \frac{2kMr}{c^2} - \frac{4\pi kQ^2}{c^4} \tag{29a}$$

$$\Sigma = r^2 + a^2\cos^2\theta \tag{29b}$$

$$\Delta = r^2 + a^2 - \Omega \tag{29c}$$

$$A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \tag{29d}$$

the presence of the cross term $(-2a\Omega/\Sigma)\sin^2\theta\, d\varphi\, cdt$ invalidates the assumption that the metric under consideration is static, i.e., equation (9). To remove this term we must thus introduce a new time differential in the rotating system. Redefinition of simultaneity (Adler et al., 1975)

$$cdt^* = cdt - \frac{a\Omega}{\Sigma - \Omega}\sin^2\theta\, d\varphi \tag{30}$$

brings equation (28) into the following form:

$$ds^2 = \left(1 - \frac{\Omega}{\Sigma}\right)c^2(dt^*)^2 - \frac{\Sigma\Delta}{\Sigma - \Omega}\sin^2\theta\, d\varphi^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 \tag{31}$$

The line element (31) and equations (20) and (24) state that in the Kerr–Newman field the total energy of a test particle measured on the rotating frame is

$$E_{K\text{-}N} = \left(1 - \frac{\Omega}{\Sigma}\right)m_0 c^2 \frac{cdt^*}{ds} \tag{32}$$

which can be expanded to second order in a/r, and reads

$$E_{K\text{-}N} \cong E_{R\text{-}N} + \frac{km_0 Ma^2\cos^2\theta}{r^3} - \frac{2\pi km_0 Q^2 a^2\cos^2\theta}{r^4c^2} \tag{33}$$

with $E_{R\text{-}N}$ given by equation (27). Since the parameter a is a measure of the angular momentum per unit mass of the source

$$Ma = \frac{-1}{c}J \tag{34}$$

J being the angular momentum of the source, we see that rotation of the source can create extra potentials of rotation besides the Newtonian one. It is noted, however, that these potentials of rotation shown in equation (33) vanish in the equatorial plane where $\theta = \frac{\pi}{2}$. Hence the planetory orbits of most of the planets in our solar system are not affected by these rotation potentials in the sense of classical mechanics.

Substituting equation (30) into equation (32), we have

$$\begin{aligned} E_{K\text{-}N} &= m_0c^2\left[\left(1 - \frac{\Omega}{\Sigma}\right)\frac{cdt}{ds} - \frac{a\Omega}{\Sigma}\sin^2\theta\frac{d\varphi}{ds}\right] \\ &= m_0c^2\left(g_{44}\frac{dx^4}{ds} + g_{43}\frac{dx^3}{ds}\right) \end{aligned} \tag{35}$$

where $x^3 = \varphi$. The term $m_0c^2g_{44}dx^4/ds$ can be interpreted as the energy measured in the coordinate time system, and the term $g_{43}m_0c^2dx^3/ds$ is due to the difference of measurements in the coordinate time system and the rotating system.

In any geometry endowed with a symmetry described by a Killing vector field ξ, motion along any geodesic whatsoever leaves constant the scalar product of the tangent vector with the Killing vector (Misner et al., 1973)

$$P_K = P^\mu \xi_\mu = g_{\mu K} P^\mu$$

$$E = m_0 c^2 g_{4\mu} \frac{dx^\mu}{ds} \tag{36}$$

The energy in the Kerr–Newmann field (35) is seen to agree with expression (36) exactly.

REFERENCES

Sokolnikoff, I. C. (1969). *Tensor Analysis*. John Wiley & Sons, New York, pp. 292–296.
Reissner, H. (1916). *Annalen der Physik*, **50**, 106.
Nordström, G. (1918). *Proceedings, Koninklijke Nederlandse Akademie van Wetenschappen*, **20**, 1238.
Kerr, R. P. (1963). *Physical Review Letters*, **11**, 237.
Newman, E. T., et al. (1965). *Journal of Mathematical Physics*, **6**, 918.
Boyer, R. H., and Lindquist., R. W. (1967). *Journal of Mathematical Physics*, **8**, 265.
Adler, R. J., Bazin, M. J., and Schiffer, M. (1975). *Introduction to General Relativity*. McGraw-Hill, New York, pp. 124.
Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*. Freeman, San Francisco, pp. 651